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Clifford algebra description of non-Abelian gauge fields

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Abstract. *After a re-analysis of the Clifford algebra structure of exterior forms, we expound the Clifford algebra approach to non-Abelian gauge fields. This method is then extended to the geometrical theory of gauge fields on principal fibre bundles. A generalised Lorentz condition arises as a condition of compatibility between local and global treatment.*

1. INTRODUCTION

Clifford algebras were discovered in the second half of the nineteenth century as a generalisation of the quaternion algebra [1] [2], but only about fifty years later, when Dirac wrote his famous equation of the relativistic electron, did they became an essential tool for the description of the physical world.

As all fundamental particles (quarks and leptons) are generally supposed to have spin $1/2$ [3], it is a remarkable fact that gauge fields interact with matter substantially only via the combination $A_\mu \gamma^\mu$ (A_μ being the potentials and γ^μ the Dirac matrices): the physics of fundamental particles seems to suggest Clifford algebra as the natural mathematical tool for the description of gauge fields.

It is classical that, exploiting the linear isomorphism between Clifford and exterior algebras [4] and using Kähler operator $\not{d} := d - \delta$ [5], it is possible to synthetize Maxwell equations in a unique relation between Clifford numbers [6, 7, 8]. In the present paper, we give a generalisation of such a local treatment to non-Abelian gauge fields [9], and

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then we extend it to the geometrical theory of these fields on principal fibre bundles. As pointed out by a referee, a brief discussion of the local theory was already given by Tucker [10].

First (sect.2) we analyse the Clifford algebra structure of exterior algebra, obtaining that a major features of Clifford product is the following one: the adjoint of the operator of left multiplication by a multivector a is the operator of left multiplication by $a^\dagger$ (where the simbol † denotes the main antiautomorphism of the exterior algebra). Moreover, we show that such a feature pertains also to the Clifford multiplication of Lie algebra valued forms, which is needed in order to describe non-Abelian gauge fields.

In section 3 we recall the local theory of gauge fields in terms of Clifford algebra, then we examine the behaviour of the theory under gauge transformations.

In section 4 we give the Clifford algebra formulation of the geometrical theory of gauge fields as differential forms on a principal fibre bundle P. We use the usual Kaluza-Klein metric of P in order to define the Clifford product of forms and the Kähler operator of P. The covariant Kähler operator is then defined as its «horizontal part». Furthermore, we write the field equation in terms of Clifford algebra and we show that the theory of sect.3 is just the local version of such a geometrical theory.

Finally we analyse the action of the covariant Kähler operator on the connection form ω, obtaining that the pull back, via a local section σ, of this action to the base manifold M coincides with the action of the covariant Kähler operator of M if and only if $\sigma^*\omega$ satisfies a generalised Lorentz condition.

2. GEOMETRIC ALGEBRAS

Let V be a vector space and g a scalar product in it; we shall admit also the case of g non definite. By $\bigwedge(V)$ we denote the exterior algebra of V. We recall that the main antiautomorphism † of $\bigwedge(V)$ is defined by

1) $a^\dagger = a$ if $a \in \bigwedge^0(V) \oplus \bigwedge^1(V)$,

2) $(a \wedge b)^\dagger = b^\dagger \wedge a^\dagger$.

We introduce the operator $I_u : \bigwedge(V) \to \bigwedge(V)$ of *left inner multiplication* by a vector $u \in V$ as the adjoint of the operator E_u of left *exterior* multiplication by u:

$$\hat{g}(a, I_u b) = \hat{g}(E_u a, b) = \hat{g}(u \wedge a, b) \ , \ \forall a, b \in \bigwedge(V) \ , \tag{1}$$

where $\hat{g}$ is the metric of $\bigwedge(V)$ induced by g. By virtue of the non degeneracy of g this equation defines uniquely I_u; in agreement with the usual notations (see e.g. ref. [7]), we shall also write $u \cdot b$ for $I_u b$. It is well known that, if we denote by J the isomorphism of $\bigwedge(V)$ with $\bigwedge(V^*)$ induced by the metric, and with $\lrcorner$ the usual interior product of a vector with a form we have

$$I_u a = J^{-1}(u \lrcorner (Ja)) \ .$$

Moreover, it is easy to prove that I_u is the unique antiderivation of $\bigwedge(V)$ having the property that

$$I_u w = g(u, w) , \tag{2}$$

for all $u, w \in V$.

It is possible to express the inner product in terms of the usual operations on exterior algebras; if $u \in V$ and $a \in \bigwedge^s(V)$ we have

$$u \cdot a = (-1)^{|g|}(-1)^{(s-1)(n-s)} *(u \wedge *a) , \tag{3}$$

where $|g|$ is the determinant of the metric of V, n is the dimension of V, and $*$ the Hodge duality operator (in order to prove this relation it is enough to evaluate the expression $\hat{g}(b, *(u \wedge *a))$).

The operation of inner multiplication can be extended to multivectors: we define $a \cdot b$ according to

$$\hat{g}(a^\dagger \wedge c, b) = \hat{g}(c, a \cdot b) \quad \forall c \in \bigwedge(V) ;$$

then, if $a \in \bigwedge^r(V)$, $b \in \bigwedge^s(V)$, with $s \geq r$, the following equation holds

$$a \cdot b = (-1)^{|g|}(-1)^{\frac{(r-1)r}{2}}(-1)^{(s-r)(n-s)} *(a \wedge *b) .$$

Now we introduce the operator L_u of *left Clifford or geometric multiplication* by a vector $u \in V$ as

$$L_u := E_u + I_u . \tag{4}$$

We shall write briefly ua for $L_u a$. From our construction it is obvious that L_u is a selfadjoint operator.

When combined via the Clifford product, the vectors generate the whole $\bigwedge(V)$; therefore we can use this property to extend Clifford multiplication to an associative product of multivectors obtaining the usual Clifford multiplication of multivectors (see *e.g.* ref. [4]). The Clifford product of multivectors will always be denoted by juxtaposition of the factors.

It is very easy to see that, as was pointed out by Kähler [11], one has

$$\hat{g}(ab, c) = \hat{g}(b, a^\dagger c) . \tag{5}$$

It is clear that in the present context *the geometric product is just the simplest modification of the exterior product, which has the associative property and satisfies* eq. (5).

Notice that by our procedure it is possible to define a geometric product in every Z_+ -graded, associative algebra, admitting a set of generators of degree one, and whose underlying vector space is endowed with a metric. It is easy to see that this geometric product is always such that the scalar part of $a^\dagger a$ is the «square of the norm» of a.

In general it is not trivial to generalise this procedure to other algebras, but it is interesting to note that Lie algebra, one of the most used algebras in physics, is in some sense a «geometric algebra». In fact, if $\mathcal{G}$ is a semisimple Lie algebra and k its Killing-Cartan form (that by Cartan's theorem is nondegenerate), then one has [12]

$$k([x,y],z) = k(y,[-x,z]) \quad \forall x,y,z \in \mathcal{G}\,; \tag{6}$$

since the operation $x \mapsto -x$ is an antiautomorphism of $\mathcal{G}$, eq. (6) is a generalisation of eq. (5) to $\mathcal{G}$ and shows that the Lie algebra product has the main feature of the geometric product.

Consider now the dual space V^* of V, and the algebra $\bigwedge(V^*,\mathcal{G})$, of $\mathcal{G}$ – valued multicovectors on V:

$$\bigwedge(V^*,\mathcal{G}) \simeq \bigwedge(V^*) \otimes \mathcal{G}\,.$$

It is well known that the exterior product of two multicovectors of this space can be defined by bilinear extension of the following product of decomposable multicovectors:

$$\alpha \wedge \beta = (a \otimes x) \wedge (b \otimes y) := (a \wedge b) \otimes [x,y]\,,$$

where $\alpha,\beta \in \bigwedge(V^*,\mathcal{G}) a,b \in \bigwedge(V^*)$, and $x,y \in \mathcal{G}$.

In an analogous way we can define the geometric product of two multicovectors as the bilinear extension of the following product:

$$\alpha\beta = (a \otimes x)(b \otimes y) := (ab) \otimes [x,y] \tag{7}$$

If we introduce in $\bigwedge(V^*,\mathcal{G})$ the metric (kg) induced by $\hat{g}$ and k, and the main antiautomorphism induced by the antiautomorphisms of $\bigwedge(V^*)$ and $\mathcal{G}$, namely the linear extensions of

$$(kg)\big(\alpha,\beta\big) = (kg)((a \otimes x),(b \otimes y)) := \hat{g}(a,b)\,k(x,y)\,,$$

$$\alpha^\dagger = (a \otimes x)^\dagger := a^\dagger \otimes (-x)\,,$$

respectively, we obtain that (7) really defines a geometric product, *i.e.*, the relation

$$(kg)\big(\alpha\theta,\beta\big) = (kg)\big(\theta,\alpha^\dagger\beta\big)$$

holds for all $\alpha, \beta, \theta \in \bigwedge(V^*, \mathcal{G})$. Notice that, if we put

$$\alpha \cdot \beta := (a \cdot b) \otimes [x, y] ,$$

then

$$(kg)(\alpha \wedge \theta, \beta) = (kg)(\theta, \alpha^\dagger \cdot \beta) .$$

3. GAUGE FIELDS, LOCAL THEORY

Let M be a four dimensional pseudo-Riemannian manifold, and $\bigwedge(T^*M)$ the exterior bundle of M . The Clifford product of two multicovectors in $\bigwedge(T^*_x M)$ will still be denoted by juxtaposition. The Kähler operator on the forms of M is then defined as

$$\not{d} : \Gamma(\bigwedge(T^*M)) \to \Gamma(\bigwedge(T^*M))$$
$$a \mapsto \not{d}a := da - \delta a ,$$

where $\Gamma(\bigwedge(T^*M))$ denotes the space of the sections of $\bigwedge(T^*M)$, d is the exterior differential and δ its adjoint operator ($\delta := (-1)^{|g|} * d*$).

If we introduce a local frame $(e^\mu)_{\mu=1,2,3,4}$, of $\Gamma(T^*M)$ and its dual frame $(e_\mu)_{\mu=1,2,3,4}$, then using eq. (4) it is easy to see that the action of $\not{d}$ on a form a is given by [11]

$$\not{d}a = e^\nu(\nabla_{e_\nu} a) . \tag{8}$$

(the Einstein summation convention has been used) where ∇_{e_ν} is the covariant derivative with respect to the Levi-Civita connection, in the direction e_ν . We point out that the existence of Kähler operator deos not require any topological condition on the manifold M ; in fact this operator is build up just using the exterior differential and the exterior codifferential, which exist on every pseudo-Riemannian manifold.

We turn now to gauge fields. It is well known that the electromagnetic field can be represented locally by a $2-$ form F on an appropriate open subset U of space-time M and that Maxwell equations can be written as

$$\not{d}F = -j , \tag{9}$$

where j is the $1-$ form representing the electromagnectic current. In fact we can split (9) into its vector and $3-$ form parts obtaining the usual Maxwell equations in terms of differential forms:

$$dF = 0 ,$$
$$\delta F = j .$$

We point out that, if in relation (9) we substitute the 1-form j with a form $\jmath \in \Gamma(\bigwedge^1 (T^*U) + \bigwedge^3(T^*U))$, we obtain a theory with magnetic monopoles [13].

In general, a gauge field can be represented locally by a section of the fibre bundle $\bigwedge(T^*U,\mathcal{G})$ of $\mathcal{G}$-valued forms on $U \subset M$, where $\mathcal{G}$ is the Lie algebra of the gauge group; it is possible to exploit the geometric product of the space of Lie-algebra valued forms in order to write the field equation of a non-Abelian gauge field in terms of Clifford algebra [10]. Indeed, consider the *local covariant Kähler operator* $\not{D}$:

$$\not{D} : \Gamma(\bigwedge(T^*U,\mathcal{G})) \to \Gamma(\bigwedge(T^*U,\mathcal{G}))$$
$$\alpha \mapsto \not{D}\alpha := \not{d}\alpha + A\alpha ,$$

where A is the $1-$form representing the potential of the gauge field, and $\not{d}$ is here the operator induced on $\Gamma(\bigwedge(T^*U,\mathcal{G}))$ by the Kähler operator, namely, it is the additive extension of the following operator acting on decomposable forms: $\not{d}(a \otimes x) := (\not{d}a) \otimes x$. Notice that the restriction of $\not{D}$ to compactly supported sections is skewsymmetric.

The field equations for the $1-$form F representing the field strength of the gauge field are given by

$$\not{D}F = -j , \tag{10}$$

where $j \in \Gamma(\bigwedge^1(T^*U,\mathcal{G}))$ is the current one-form. By the way, we point out that, in order to obtain a theory with non-Abelian magnetic monopoles, it is enough to suppose that $j \in \Gamma(\bigwedge^1(T^*U,\mathcal{G}) + \bigwedge^3(T^*U,\mathcal{G}))$.

Using (4), it easy to see that equation (10) is equivalent to

$$DF = 0 \tag{11}$$

$$\delta^A F = j . \tag{12}$$

where we have denoted by $DF := dF + A \wedge F$ the covariant differential of F and by $\delta^A F := \delta F - A \cdot F$ the covariant codifferential. Equation (11) represents the «Bianchi identity» satisfied by F, while (12) is the inhomogeneous field equation.

Notice that, applying $\not{D}$ to (10), one has

$$(\not{D})^2 F = \not{D} j ;$$

the scalar part of this equation is

$$\delta^A j = 0 ,$$

namely, the equation of current conservation.

Moreover, if, as is natural in this context, in order to satisfy (11) we put

$$\not{D}A = F\ ,$$

the generalised Lorentz condition

$$\delta^A A = 0 \tag{13}$$

follows.

Now we shall examine the behavior of this theory under gauge transformation; for simplicity we shall suppose that the group G is a matrix group (in order to generalise what follows to a generic Lie group it would be enough just to use a little heavier notation) and, in order to clarify the situation we shall use explicitly a coordinate frame. Under gauge transformation the potentials transform according to

$$A' = \tau^{-1} A \tau + \tau^{-1} d\tau\ .$$

where $\tau : U \rightarrow G$ is the element by which the gauge transformation is performed, and we have written $d\tau$ for τ_*. The field strength transforms according to the adjoint representation of G:

$$F' = \tau^{-1} F \tau\ .$$

Then we have

$$\begin{aligned} \not{D}' F' &= dx^\mu \nabla_{\partial/\partial x^\mu}(\tau^{-1} F \tau) + A' F' = \\ &= \tau^{-1} \not{d} F \tau + \tau^{-1} A F \tau - dx^\mu (dx^\nu \wedge dx^\rho) \\ &\otimes [\tau^{-1} \partial_\mu \tau, F'_{\nu\rho}] + \tau^{-1} d\tau F' = \tau^{-1} (\not{D} F) \tau\ . \end{aligned}$$

Notice that, if we want that $\not{D}' A' = F'$, we have to restrict the allowed gauge transformation in such a way that A' satisfies (13).

4. GAUGE FIELDS, GLOBAL THEORY

We recall that, from the geometrical point of view, the gauge field strength is a $2-$form on a principal fibre bundle P having a compact, semisimple Lie group G as structural group and the space-time M as base manifold; moreover, P is endowed with a connection form ω [14]. Here we shall extend the Clifford algebra treatment of gauge field to this geometrical theory and we shall discuss some consequences.

The metric h of P is defined by

$$h_p(X,Y) = g_{\pi(p)}(\pi_* X, \pi_* Y) + k(\omega_p(X), \omega_p(Y)) \ ,$$

where k is the Killing Cartan metric of G, π the canonical projection of P, and $X,Y \in T_pP$ [15] [16].

Now, acting exactly in the same way as in section 2, we can define the geometric multiplication of forms, and extend this operation to $\mathcal{G}$ – valued exterior forms on P. Remark that the Clifford algebra structure of $\bigwedge(T^*P)$ (and of $\bigwedge(T^*P,\mathcal{G})$) depends on the connection throught the metric.

Since the horizontal and vertical subspaces are perpendicular, we have that, if a_1 and a_2 are horizontal (vertical) multicovectors, namely $a_i(X_1, ..., X_{k_i}) = 0$ when one of the $X_l \in TP$ is vertical (horizontal), then $a_1 a_2$ too is horizontal (vertical); on the other side, if a_1 is horizontal and a_2 is vertical, (or *viceversa*), one has

$$a_1 a_2 = a_1 \wedge a_2 \ .$$

Consider now the fibre bundle $\bigwedge_H(T^*P)$ of the horizontal forms on P; we remark that, not only the set of forms constituting this space is independent of the connection, but also *its Clifford algebra structure is independent of the connection.* In fact, by virtue of the definition of h, $\bigwedge_H(T_p^*P)$ is canonically isomorphic to $\bigwedge(T_{\pi(p)}M)$, *as a metric space* and therefore also as a Clifford algebra. Moreover, since for every local section σ of P the restriction of the pull-back application to this space, $\sigma^*|_{\bigwedge_H(T^*P)}$, is an isometry, it is clear that it is also a Clifford algebra homomorphism. On the other side, if a is a vertical form, it is not true that $(\sigma^* a)(\sigma^* b) = \sigma^*(ab)$.

The Kähler operator $\not{d}$ of P is defined according to

$$\not{d} := d - \delta \ ;$$

then, in every reference frame $(x^A)_{A=1,...,4+n}$, it is given by

$$\not{d} = dx^A \nabla_{\partial/\partial x^A} \ ,$$

where ∇ is the covariant derivative with respect to the Levi Civita connection of h.

The *covariant* Kähler operator on $\bigwedge(T^*P,\mathcal{G})$ can be defined as

$$\not{\mathcal{D}} : \Gamma\left(\bigwedge(T^*P,\mathcal{G})\right) \to \Gamma\left(\bigwedge(T^*P,\mathcal{G})\right) \qquad (14)$$

$$\alpha \mapsto \not{\mathcal{D}}\alpha := (\not{d}\alpha)^H \ ,$$

where the index H denotes the operation of taking the horizontal part. We have

$$\not{\mathcal{D}}\alpha = (d\alpha)^H - (\delta\alpha)^H = \mathcal{D}\alpha - \delta^\omega \alpha \ ,$$

where $\mathcal{D}$ is the usual covariant differential, and $\delta^\omega\alpha := (\delta\alpha)^H$ the covariant codifferential; notice that $\delta^\omega \neq \pm *\mathcal{D}*$. Now we study some properties of these operators.

First we observe that the restriction of $\not{\mathcal{D}}$ to the space of compactly supported sections of $\bigwedge_H(T^*P,\mathcal{G})$ (horizontal $\mathcal{G}-$valued forms) is skewsymmetric; in fact, if α, β are compactly supported forms of $\Gamma(\bigwedge_H(T^*P,\mathcal{G}))$, we have

$$\int_P (kh)(\not{\mathcal{D}}\alpha,\beta)\mu_P = \int_P (kh)(\not{d}\alpha,\beta)\mu_P = \int_P (kh)(\alpha,-\not{\mathcal{D}}\beta)\mu_P ,$$

where (kh) is the metric induced by h and k on $\bigwedge(T^*P,\mathcal{G})$, and μ_P is the volume form of P induced by h. Moreover, the restriction of δ^ω to compactly supported sections of $\bigwedge_H(T^*P,\mathcal{G})$ is the adjoint operator of the restriction of $\mathcal{D}$ to the same space:

$$\begin{aligned}\int_P (kh)(\mathcal{D}\alpha,\beta)\mu_P &= \int_P (kh)(d\alpha,\beta)\mu_P = \int_P (kh)(\alpha,\delta\beta)\mu_P \\ &= \int_P (kh)(\alpha,\delta^\omega\beta)\mu_P ,\end{aligned} \tag{15}$$

for all compactly supported sections α,β of $\bigwedge_H(T^*P,\mathcal{G})$. Notice that, in order to obtain relation (15), it is enough to suppose that only one of the forms α and β has compact support.

Let us consider the space $\Gamma_{Ad}(\bigwedge(T^*P,\mathcal{G}))$ of the forms $\alpha \in \Gamma(\bigwedge_H(T^*P,\mathcal{G}))$ that satisfy

$$R_g^*\alpha = Ad_{g^{-1}*}\alpha ; \tag{16}$$

since the connection form too satisfies relation (16), this space is invariant under the action of $\not{\mathcal{D}}$.

We shall show that for $\alpha \in \Gamma_{Ad}(\bigwedge(T^*P,\mathcal{G}))$, we have

$$\sigma^*(\not{\mathcal{D}}\alpha) = \not{\mathcal{D}}(\sigma^*\alpha) \tag{17}$$

for all local section $\sigma : U \to P$ $(U \subset M)$. First we recall that, if $\alpha \in \Gamma_{Ad}(\bigwedge(T^*P,\mathcal{G}))$, we have $\mathcal{D}\alpha = d\alpha + \omega \wedge \alpha$ and therefore

$$\sigma^*(\mathcal{D}\alpha) = D(\sigma^*\alpha) ,$$

where the covariant derivative « D » is calculated with respect to the potential $A = \sigma^*\omega$.

Let β be an element of $\Gamma_{Ad}(\bigwedge(T^*P,\mathcal{G}))$ with compact support contained in $\pi^{-1}(U)$; then, since σ^* is an isometry from the space $\bigwedge_H(T^*_{\sigma(x)}P,\mathcal{G})$ to the space

$\bigwedge(T_x^*M,\mathcal{G})$, and the functions $(kh)(\delta^\omega\alpha,\beta)$ and $(kh)(\alpha,\mathcal{D}\beta)$ are constants along the fiber of P, we have

$$\int_U (kg)(\sigma^*(\delta^\omega\alpha),\sigma^*\beta)\mu_M = \frac{1}{\mathrm{Vol}(G)}\int_{\pi^{-1}(U)}(kh)(\delta^\omega\alpha,\beta)\mu_P =$$

$$= \frac{1}{\mathrm{Vol}(G)}\int_{\pi^{-1}(U)}(kh)(\alpha,\mathcal{D}\beta)\mu_P =$$

$$= \int_U (kg)(\sigma^*\alpha, D(\sigma^*\beta))\mu_M = \int_U (kg)(\delta^{\sigma^*\omega}(\sigma^*\alpha),\sigma^*\beta)\mu_M ,$$

where μ_M is the volume form of M and Vol (G) the volume of the structure group; then (17) follows.

Let us consider the action of the covariant Kähler operator on vertical forms satisfying condition (16); we show below that $\not{\mathcal{D}}\alpha = \mathcal{D}\alpha$.

Notice that, since α satisfies equation (16), we have $(\delta\alpha)^H \in \Gamma_{Ad}(\bigwedge(T^*P,\mathcal{G}))$. Choosing a basis $(\xi_a)_{a=1\dots n}$ in $\mathcal{G}$, we have $\alpha = \alpha^a \otimes \xi_a$, with $\alpha^a \in \Gamma(\bigwedge(T^*P))$. If $\beta = \beta^a \otimes \xi_a$ is a compactly supported form of $\Gamma_{Ad}(\bigwedge(T^*P,\mathcal{G}))$, then

$$\int_P (kh)((\delta\alpha)^H,\beta)\mu_P = \int_P k_{ab}\beta^b \wedge *\delta\alpha^a = \int_P k_{ab}d\beta^b \wedge *\alpha^a$$

$$= \int_P (kh)(\alpha, d\beta)\mu_P ,$$

where $k_{ab} = k(\xi_a,\xi_b)$, therefore we obtain

$$\int_P (kh)((\delta\alpha)^H,\beta)\mu_P = \int_P (kh)(\alpha,\mathcal{D}\beta - \omega\wedge\beta)\mu_P =$$

$$= \int_P (kh)(\alpha,-\omega\wedge\beta)\mu_P = \int_P (kh)(\omega\cdot\alpha,\beta)\mu_P$$

and so, by virtue of the arbitrariness of $\beta \in \Gamma_{Ad}(\bigwedge(T^*P,\mathcal{G}))$, we obtain

(18) $$\delta^\omega\alpha = (\omega\cdot\alpha)^H = 0 .$$

In particular, we have

(19) $$\delta^\omega\omega = 0 .$$

Now we come to gauge fields. It is well known that the field strength is mathematically represented by a horizontal $2-$form $\mathcal{F} \in \Gamma_{Ad}(\bigwedge(T^*P,\mathcal{G}))$; then the equations of motion for $\mathcal{F}$ can be written in the form

(20) $$\not{\mathcal{D}}\mathcal{F} = -\mathcal{J} ,$$

where $\mathcal{J}$ is the $1-$ form representing the current. Equation(20) can be split into

$$\mathcal{D}\mathcal{F} = 0 \ , \tag{21}$$

$$\delta^{\omega}\mathcal{F} = \mathcal{J} \ . \tag{22}$$

The first of these equations represents the Bianchi identity for $\mathcal{F}$. Equation (17) shows that *this is exactly the global version of the local theory developed in the previous section.*

We remark that, in virtue of (19), eq. (21) can be solved by putting

$$\mathcal{F} = \not{\mathcal{D}}\omega \ .$$

We point out that the generalised Lorentz condition considered in choosing the gauge becomes here a compatibility condition between the covariant Kähler derivative of the connection and its local representations; in fact, one has that

$$\sigma^*(\not{\mathcal{D}}\omega) = \not{\mathcal{D}}(\sigma^*\omega)$$

holds only if σ is such that $A = \sigma^*\omega$ satisfies the generalised Lorentz condition (13).

Finally consider a vertical automorphism f of the bundle P, namely a gauge transformation; since equation (19) is a consequence of the verticality of the connection form, it is satisfied also by the transformed connection $f^*\omega$; but, in general, we have that $\sigma^*(f^*\omega)$ does not satisfy the Lorentz condition even if $\sigma^*\omega$ does. The theory on the fibre bundle is thus invariant under the whole gauge group, while the natural invariance group of the theory on the basis is the subgroup of the gauge group leaving (13) invariant.

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REFERENCES

[1] W.K. CLIFFORD: *Application of Grassmann's extensive algebra*, Am. J. Math., 1, (1878), p. 350.

[2] W.K. CLIFFORD: *On the Classification of Geometric Algebras*, paper XLIII in *Mathematical papers of W.K. Clifford*, edited by R. Tucker Mac Millan, London, 1882.

[3] See whatever text on the classification of elementary particles in terms of quarks, or on Standard Model; *e.g.*, C. Quigg: *Gauge Theories of the Strong, Weak and Electromagnetic Interactions*, Benjamin/Cummings Inc., 1983.

[4] See *e.g.* C. CHEVALLEY: *The Algebraic Theory of Spinors*, Columbia University Press, 1954.

[5] As pointed out by a referee, this operator has been used for the first time by Landau and Ivanenko in order to describe the «Magnetischen Electron». See *Zur Theorie des Magnetischen Electron. I.*, Z. Physik, **48**, (1928), p. 340.

[6] M. RIESZ: *Clifford Numbers and Spinors*, Institute for Fluid Dynamics and Appl. Math. No. 38, University of Maryland, College Press, Md., 1958.

[7] See also D. HESTENES: *Space-Time Algebra*, Gordon and Breach, New York, 1966.

[8] Laporte and Uhlenbech gave a different «spinor» description of electromagnetic field using «two spinors»; see *Application of spinor analysis to the Maxwell and Dirac equations*, Phys. Rev., **37**, (1931), p. 1380.

[9] The «two spinor» formalism has been generalised to non-Abelian gauge fields in R. Penrose - W. Rindler: *Spinors and Space-Time*, Vol. 1, Cambridge University Press, 1984.

[10] R.W. TUCKER: *A Clifford calculus for physical field theories*, in «Clifford Algebras and their Applications in Mathematical Physics», edited by J.S.R. Chisholm and A.K. Common, NATO ASI series vol. 183, 1985, p. 177.

[11] E. KÄHLER: *Der innere Differentialkalkül*, Rendiconti di Matematica (5), **21**, (1962), p. 425.

[12] See *e.g.* N. Jacobson: *Lie Algebras*, Interscience Publisher, New York, 1962.

[13] M.A. DEFARIA-ROSA, E. RECAMI, W.A. RODRIGUES JR.: *A satisfactory formalism for magnetic monopoles by Clifford algebras*, Phys. Lett. B, **173**, (1986), p. 233.

[14] For the geometric theory of gauge fields see D. Bleecker: *Gauge Fields and Variational Principles*, Addison-Wesley Publishing Inc., Massachuset, 1981.

[15] See *e.g.*, Y.M. CHO: *Higher-dimensional unification of gravitation and gauge theories*, J. Math. Phys., **16**, (1975), p. 2029.

[15] R. KERNER: *Geometrical background for the unified field theories: the Einstein Cartan theory over a principal fibre bundle*, Ann. Ins. Henri Poincaré, **34**, No. 4, (1981), p. 437.

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